Comment on "Coupled dynamics of atoms and radiation–pressure–driven interferometers"

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In two recent articles [1, 2], Meiser and Meystre describe the coupled dynamics of a dense gas of atoms and an optical cavity pumped by a laser field. They make two important simplifying assumptions: (i) *the gas of atoms forms a regular lattice and can be replaced by a fictitious mirror*, and (ii) *the atoms strive to minimize the dipole potential*. We show that the two assumptions are inconsistent: the configuration of atoms minimizing the dipole potential is not a perfect lattice. Assumption (ii) is erroneous, as in the strong coupling regime the dipole force does not arise from the dipole potential. The real steady state, where the dipole forces vanish, is indeed a regular lattice. Furthermore, the bistability predicted in [1, 2] does not occur in this system.

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In two recent articles [1, 2], Meiser and Meystre describe the coupled dynamics of movable atoms, mirrors and light in a pumped optical cavity. They take the effect of the atoms on the light field into account using a 1-dimensional non-perturbative model introduced by Deutsch et al. [3]. The mechanical effects of the off-resonant light on the atoms are described by a dipole Hamiltonian. They claim that the atoms trapped in the cavity field self-organize to form a regular lattice of atom clouds which behaves as a beam splitter (BS). This BS effectively splits the single cavity into two coupled resonators ("left" and "right"). An important prediction of [1, 2] is a bistability effect: optical forces will push this "atom BS" to a position where it is approximately an integer multiple of the half-wavelength away from the left (right) cavity mirror. Thus for a certain parameter regime the "left" ("right") cavity is on resonance with the pump laser, and has intense light, while the other cavity has weak field.

We argue that the model used by Meiser and Meystre contains contradictions and errors involving the way the dynamics of the atoms (taking place on a much shorter timescale than that of the movable cavity mirror in [1]) is treated. The model hinges on two key assumptions. The first is a claim based on [3]: (i) *the gas of atoms forms a regular lattice of pancake-shaped clouds with lattice constant* $d_0 = \lambda/2(1 + 2\,\mathrm{atan}\Lambda/\pi)$*, and can thus be replaced by a fictitious "atom BS"*. Here $\Lambda = k\eta\alpha/(2\epsilon_0)$ is the dimensionless polarizability density of a cloud of surface density η composed of atoms of polarizability $\alpha < 0$, with $k = 2\pi/\lambda$ denoting the free-space wavenumber of the pump laser. The second assumption is used to find the steady state position of the atom BS: (ii) *the atoms strive to minimize the total dipole potential* H_{int} *(eq. (2) of [1], or eq. (3) of [2])*. In this Comment we show that these assumptions are inconsistent, and that (ii) has to be replaced by a formula for the *force* on the atoms. We present the required expression for the force which allows us to determine the steady state of this system. This does not exhibit the bistability phenomenon of [1, 2].

The setup considered in [1, 2] is an open system: laser light enters the cavity through the mirrors, carrying momentum and energy, and is coupled out of the cavity at the mirrors. It is not at all clear what function of the system parameters is minimized in a steady state. Minimizing the dipole energy certainly does not lead to steady state configurations. To illustrate this point, it is worthwhile to consider a conceptually simpler situation, namely atoms trapped in a standing wave laser field without any cavities involved. As illustrated in Fig. 1, using the model of Meiser and Meystre but allowing all of the atom clouds to move independently, a Monte Carlo algorithm to minimize the dipole energy leads to configurations differing from a simple lattice. Starting from the regular lattice configuration, the atoms decrease the dipole energy by forming a *self-organized cavity* resonant with the pump beam. The intensity inside this cavity is thus enhanced by the resonance, and a few atoms coupled to this intense field contribute to the total dipole potential by such a large amount, that its absolute value exceeds that of the original energy by orders of magnitude (see the figure caption for details). The mirrors of this self-organized cavity are slabs of an atomic lattice with lattice constant exceeding d_0 such that the pump field is

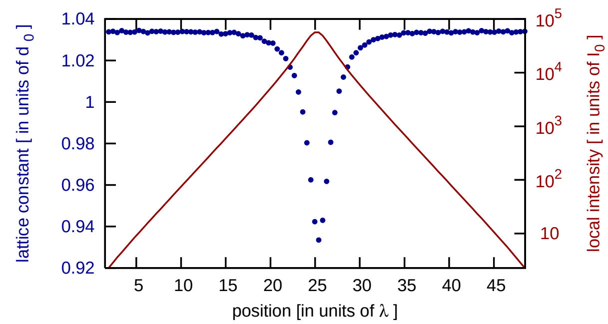

FIG. 1: (color online) A configuration found by Monte-Carlo type minimization of the dipole potential, for $N = 100$ atom clouds with polarizability $\Lambda = 0.1$, symmetrically pumped in free space (no cavity). The local lattice constant (blue dots, in units of the original $d_0 = 0.468\lambda$) and the electric field intensity at the atomic positions (red line, exponential scale, in units of the pump intensity I_0) are plotted. The total dipole energy is some 1000 times lower than that of the regular optical lattice of assumption (i). The atoms form two slabs with lattice constant exceeding d_0, impervious to the pump laser (pump frequency deep in the band gap, see [3]). These two self-organized atom mirrors constitute a high-Q cavity, trapping light in the middle. There the intensity is some 10^4 times that of the pump lasers. Inside the "atom mirrors" the intensity falls off exponentially.

in the band gap [3] of these slabs and decays exponentially inside them. The complicated spatial structure of this solution clearly shows that assumption (ii) cannot lead to (i), one of them has to be dropped. The configuration represented in Fig. 1 is not a steady state at all: as we discuss below, radiation pressure would push the two slabs apart. Thus, assumption (ii) has to be revisited. We remark that inside an optical cavity, dipole energy minimization leads to similar artifacts, in the "superstrong coupling" limit as well.

The reason why dipole energy minimization does not supply the steady states is that the atomic positions are coupled parametrically to the light field. This problem is met, and is tackled in a very neat way, when the motion of atoms coupled to a *single-mode high-Q* optical resonator is to be described in a classical approximation (Ehrenfest theorem). In the standard approach (e.g., [4]), the starting point is the quantum Hamiltonian $\hat{H} = \sum_{j=1}^{N} \hat{H}_{\rm atom}(\hat{p}_j, \hat{\sigma}_j, \hat{\sigma}_j^\dagger) + \hat{H}_{\rm field}(\hat{a}, \hat{a}^\dagger) + \hbar \sum_{j=1}^{N} g(\hat{z}_j)(\hat{\sigma}_j^\dagger \hat{a} + \hat{\sigma}_j \hat{a}^\dagger)$, with $\hat{z}_j$, $\hat{p}_j$, and $\hat{\sigma}_j$ denoting the position, momentum, and deexcitation operator of the j-th atom, $g(z)$ the mode function of the cavity, and $\hat{a}$ the cavity photon annihilation operator. The coupling between atoms and the cavity field is of the celebrated Jaynes–Cummings type. The classical approximation should furnish equations of motion for $z_j = \langle \hat{z}_j \rangle$ and $p_j = \langle \hat{p}_j \rangle$. In order to derive these, for slowly moving atoms, the separation of the timescales is invoked. The internal variables $\hat{a}$ and $\hat{\sigma}_j$, which equilibrate fast on the timescale of atomic motion, are replaced by their adiabatic steady state expectation values, which depend on all of the atom coordinates $\hat{z}_l$ (as well as the intensities and phases of the pumping lasers). One could be tempted to use the "effective" Hamiltonian $\hat{H}_{\rm eff}(\hat{z}_j, \hat{p}_j)$ obtained from $\hat{H}$ in this way, and derive the atomic dynamics from it via the Heisenberg equations of motion, i.e., $dp_j/dt = \langle -d\hat{H}_{\rm eff}/d\hat{z}_j \rangle = -d\langle \hat{H}_{\rm eff}\rangle/dz_j = -d\langle \hat{H}\rangle/dz_j$. The correct procedure, however, is to apply the adiabatic approximation to the original Heisenberg equations

$$\frac{d}{dt}\hat{p}_j = \frac{1}{i\hbar}[\hat{p}_j, \hat{H}] = -\frac{d}{d\hat{z}_j}\hat{H} = \hat{F}_j, \tag{1}$$

where the force operator is $\hat{F}_j = -\hbar(\hat{\sigma}_j^\dagger \hat{a} + \hat{\sigma}_j \hat{a}^\dagger)\, dg(\hat{z}_j)/d\hat{z}_j$, and use $dp_j/dt = \langle \hat{F}_j \rangle$. In other words, the differentiation should only be applied w.r.t. *explicit* z_j-dependence of $\langle \hat{H} \rangle$, since $\langle d\hat{H}/d\hat{z}_j \rangle$ is not the same as $d\langle \hat{H}\rangle/dz_j$. Using $d\langle \hat{H}\rangle/dz_j$ to define the dynamics supplies steady states where the "dipole potential" $\langle \hat{H} \rangle = \langle \hat{H}_{\rm eff} \rangle$ is minimized, but these are not the true steady states of the system: in these states the force on the atom $\langle \hat{F}_j \rangle$ does not vanish.

We now turn to the setup considered by Meiser and Meystre, where the back-action of the atoms on light is so substantial that the cavity no longer has a fixed mode function. To be self-contained, and to fix notation, we briefly summarize the model, detailed in [1, 2, 3].

Assuming that the atoms are fixed on the timescale of the field dynamics, the light field is calculated by solving the Helmholtz equation in a one-dimensional approximation,

$$\partial_z^2 E(z) + k^2 E(z) = -2k\Lambda E(z) \sum_j \delta(z - z_j). \tag{2}$$

The right-hand-side embodies the polarizability of the trapped atoms, which are assumed to form pancake-shaped clouds of axial size much smaller than a wavelength. The solution of this equation is trivial: between two atom clouds, the electric field is a superposition

$$\begin{aligned} E(z_{j-1} < z < z_j) &= A_j e^{ik(z-z_j)} + B_j e^{-ik(z-z_j)} \\ &= C_{j-1} e^{ik(z-z_{j-1})} + D_{j-1} e^{-ik(z-z_{j-1})}. \end{aligned} \tag{3}$$

The field has to fulfil boundary conditions:

$$E(z = z_j - 0) = E(z = z_j + 0); \tag{4a}$$

$$\partial_z E(z = z_j - 0) = \partial_z E(z = z_j + 0) + 2k\Lambda E(z_j). \tag{4b}$$

These conditions are equivalent to representing the atom clouds by BS's, i.e., $A_j = tC_j + rB_j$, $D_j = tB_j + rC_j$ with complex reflection and transmission coefficients $r = i\Lambda/(1 - i\Lambda)$, $t = 1/(1 - i\Lambda)$ [3].

The dynamics of the atoms is given by the *dipole force* acting on them. Instead of minimizing a dipole potential, the true *steady state* of the system is then specified by the positions of all the atom clouds $z_j, j = 1, \ldots, N$ such that the optical field of the cavity – the solution of (2) – exerts no net force on any of the clouds. In the following we show two ways to calculate this force acting on an infinitely thin atom cloud (BS).

The force on an atom cloud is can be obtained by integrating the the force on a single atom over the whole cloud. A microscopic model of light-matter interaction leads to two types of force: the dispersive *dipole force* and the dissipative *scattering force* [5]. This latter is often referred to as "radiation pressure", but following Meiser and Meystre we use this term to denote the mechanical effects of light in general. In [1, 2] the atom-pump detuning is assumed to be so large that the scattering force can be neglected, tantamount to assuming $\Lambda \in \mathbb{R}$. For linearly polarizable particles the dipole force time-averaged over an optical period is given [5] $F = \frac{1}{4}\alpha \nabla |E(\mathbf{x})|^2$. Calculating this force for an infinitely thin disk-shaped atom cloud poses a problem, as the electric field $E(z)$ is not differentiable at the atomic positions z_j. One must calculate the force on a disk of finite extent $z_j - w \ldots z_j + w$, and only then take the limit $w \to 0$. Since the electric field is polarized in the plane of the disk, there is no surface contribution [6], and the integral in the limit of vanishing width gives

$$F_j = \frac{\eta\alpha}{8}\left(\partial_z |E|^2 (z_j - 0) + \partial_z |E|^2 (z_j + 0)\right) \tag{5}$$

for the force on a unit surface ("radiation pressure"). This result is independent of the way in which the limit is approached, i.e. of the axial density distribution of the cloud. Substituting the modal decomposition of Eq. (3), using the BS relations, and the fact that as $\Lambda \in \mathbb{R}$, we have $|A_j|^2 + |D_j|^2 = |B_j|^2 + |C_j|^2$, some algebra leads to the simple formula

$$F_j = \frac{\epsilon_0}{2}\left(|A_j|^2 + |B_j|^2 - |C_j|^2 - |D_j|^2\right). \tag{6}$$

There is another, macroscopic way to arrive at the light-induced force on a scatterer. This consists of calculating the Maxwell stress tensor and integrating it on an arbitrary fictitious surface enclosing the body (see, e.g., Ref. [7]). In the 1D model of [1, 3], this is very easily done. For a selected atom cloud, we take the surface around it to consist of two planes orthogonal to the cavity axis, between the atom cloud and the two neighbouring clouds. As the electromagnetic wave inside the cavity is transverse, both the $\mathbf{E}$ and $\mathbf{B}$ vectors lie in the planes, and the only part of the stress tensor contributing to the integral is the term with the energy density. For the plane waves of (3) this results in (6). We remark that this line of thought is also alluded to by Meiser and Meystre, and although is not applied to the atoms, it is used to derive the force on the movable cavity mirror in Eq. (10) of [1].

The difference between (a) minimizing the dipole potential, and (b) requiring the dipole force to vanish is illustrated in Fig. 2. Here we put a weakly reflective BS ($\Lambda = 0.1$) in a high-Q cavity composed of two Dirac-δ distributions of polarizability with $\Lambda = 10$, corresponding to transmission probability $T \approx 0.01$, as in [1, 2]. To a good approximation the cavity only supports a single $\sin(z)$-mode, and since the BS is weakly reflective, we are not in the "superstrong coupling" limit: the Hamiltonian approach would constitute a reasonable approximation. Note however, that in Fig. 2(a), the intensities to the right of the trapped atom are slightly higher than to its left, indicating the corrections to the single-mode approximation, and also the bistability of [1]. When minimizing the dipole energy (a), the BS occupies a position where it is not coupled strongly to the cavity, far from an antinode of the mode function. Thus the system is near-resonant with the driving field, and the intracavity intensity is enhanced by the resonance, some 500 times the intensity outside. Extremely high intensity means large negative dipole energy for the trapped atom – however, note that at the position of the atom, the derivative of the intensity is nonvanishing, and thus the force F is nonzero. Requiring this *force* to vanish brings the atom to a mode function antinode (b), where the coupling is stronger, hence the frequency shift is larger, than in (a). For this specific example, this is already enough to shift the system out of resonance and to decrease the intracavity intensity below the free-space value.

The BS used to model the atom cloud in the cavity considered by [1, 2], has an effective $\Lambda \approx 1$, and thus the system is in the "superstrong coupling limit". We plot the force exerted by the cavity field on such a BS as a function of its position and of the detuning (equivalently, cavity size) in Fig. 3. The boundaries between the gray shaded and white areas correspond to equilibrium. Note however, that for a fixed drive detuning, of the two equlibrium solutions per half wavelength only one is stable, the one where $\partial/\partial z_a F < 0$. Thus there is no bistability of the kind predicted in [1]. The areas enclosed by the solid contour lines, where the force on the BS becomes very large, indicate that the system is on resonance and the field gradient at the BS position is high. These correspond to the black structures on Fig. 2. of [2], where the determinant D (Eq. (12) of [2]) becomes small, which there are falsely interpreted as equilibrium positions.

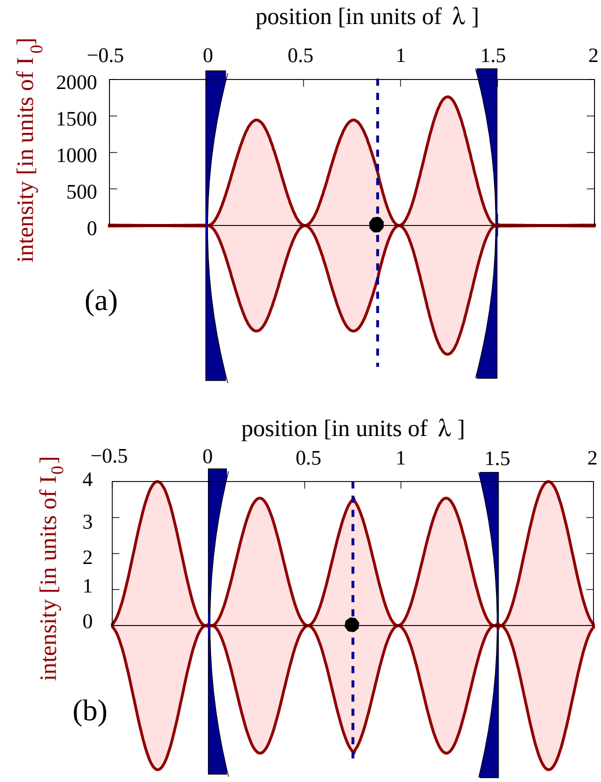

FIG. 2: (color online) Steady states of a single atom (atom cloud) of polarizability $\Lambda = 0.1$ in a symmetrically pumped cavity, according to (a) minimization of the dipole energy; and (b) vanishing dipole force. The cavity is formed by two highly reflective mirrors ($\Lambda = 10$) at $z = 0$ and $z = 1.501\lambda$. The position of the atom (black dot and vertical dotted line) induces no substantial change of the *mode function* of the cavity. However, it influences the *intensity* of the cavity mode (filled red curve, in units of the free-space intensity I_0, mirrored for better visibility). In (a), the system is on resonance, the intracavity intensity is so high (some $500 I_0$) that the intensity outside the cavity is hardly seen on this range; in (b), the atom is maximally coupled to the cavity, shifts it out of resonance and reduces the intensity below the free-space value.

In the example shown in Fig. 2, the maxima of the intensity to the left and to the right of the trapped BS are equal. In fact, an analogous statement holds for the steady state of any one-dimensional system composed of N consecutive beam splitters, held together by the dipole force, regardless of the BS parameters of the system components. This follows from formula (6) for the dipole force, whereby for every $j = 1, \ldots, N$: $|A_j|^2 + |B_j|^2 = |C_j|^2 + |D_j|^2$. Furthermore, since there is no absorption, $|A_j|^2 + |D_j|^2 = |B_j|^2 + |C_j|^2$ at every BS. These relations imply that $|A_1| = |A_2| =, \ldots, = |A_N| = |C_1| = |C_2| =, \ldots, = |C_N|$; $|B_1| = |B_2| =, \ldots, = |B_N| = |D_1| = |D_2| =, \ldots, = |D_N|$, i.e., the plane waves pass the atom clouds unattenuated, suffering only phase shifts. Thus, the envelope of the intensity oscillations of the electric field is constant throughout the sample. This is in line with the intuitive picture of "radiation pressure" caused by the collisions

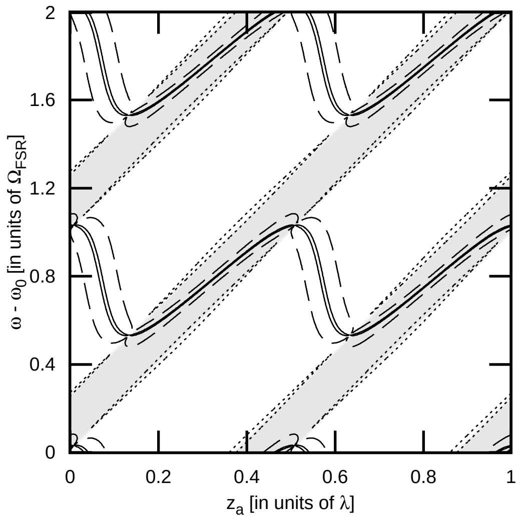

FIG. 3: Force on a single beam splitter with $\Lambda = 1$ in a standing-wave cavity with parameters as in [1, 2] (mirror transmission probability $T \approx 0$) pumped by a laser via one of the end mirrors. Contour lines are: solid, $|F| = 10F_0$; dashed, $|F| = F_0/10$; dotted, $|F| = F_0/1000$, where F_0 is the radiation pressure force that would act on the beam splitter in the absence of the cavity. Gray background indicates positive (rightward) forces, white background negative (leftward) forces. Extremely high forces ($|F| > 10F_0$, the small areas enclosed by the solid contour lines) occur whenever a part of the cavity is on resonance with the drive. The boundaries between the gray and white areas, where $F \approx 0$, are the (stable or unstable) equilibrium positions.

of photons with the atom clouds. However, these results are in direct contradiction to those obtained by Meiser and Meystre: the nonconstant field envelopes in the "bistability regime" are explicitly plotted in Fig. 3 of [2].

We now revisit assumption (i). Assuming that there exists a steady state of N identical disk-shaped atom clouds trapped by the light field in a cavity, we find that these clouds have to form a perfect lattice. This is true because in the steady state the light permeates the stack of clouds unattenuated. Thus $|E(x)|^2 = |E_0|^2 + |E_1|^2 + 2\,|E_0 E_1| \cos(2kx - \Phi(x))$ everywhere in the sample, the clouds only contribute to the phase: $\Phi(x_j < x < x_{j+1}) = \sum_{l=1}^{j} \chi_l$. The phase slip at the l'th cloud χ_l depends not only on the polarizability density Λ of the clouds, but also on the pump asymmetry, i.e., the ratio of the intensities of the left- and rightwards propagating waves, see [8] for details. Since the clouds are identical, and the light fills the structure unattenuated, both these parameters are equal for all clouds. Thus, $\chi_l = \chi$ for every $l = 1, \ldots, N$, therefore the atom clouds form a perfect lattice (possibly with gaps of an integer multiple of $\lambda/2$) with lattice constant $d = \frac{\lambda}{2\pi}(\pi - \chi)$. However, the value of the phase slip χ, and thus of the lattice constant d, is not trivial to determine, due to the dependence on the pump asymmetry. For the atoms trapped inside the asymmetrically pumped cavity considered by Meiser and Meystre, this asymmetry varies with the pump detuning: on resonance, it is negligible, while far from resonance, it is substantial. Thus Eqs. (24) and (25) of [1] (taken from [3] for symmetric pumping) *cannot be applied* to this system. Even more crucially, whether or not the equilibrium sets in depends on the *dynamics*. We have found [8] that for large lattices, even a small pump asymmetry can lead to a dynamical instability of the equilibrium configuration.

Common wisdom holds that "dipole force is conservative". Even in the strong coupling regime of cavity QED it is often possible to construct a "potential" by integrating the dipole force (e.g., [4]). For many particles trapped in the same cavity, this potential should also include the field-mediated (parametric) interaction between those particles. We have found [8], that in the generic case even this approach breaks down. For asymmetric pumping (as in [1, 2]), $\partial F_j(z_1, \ldots, z_N)/\partial z_l \neq \partial F_l(z_1, \ldots, z_N)/\partial z_j$, and thus no potential function $V(z_1, \ldots, z_N)$ can be constructed that obeys the Young theorem about the commutativity of partial derivatives.

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